Sets of Nonnegative Matrices with Positive Inhomogeneous Products*

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ABSTRACT

Let X be a set of $k\times k$ matrices in which each element is nonnegative. For a positive integer n, let $P(n)$ be an arbitrary product of n matrices from X, with any ordering and with repetitions permitted. Define X to be a primitive set if there is a positive integer n such that every $P(n)$ is positive [i.e., every element of every $P(n)$ is positive]. For any primitive set X of matrices, define the index $g(X)$ to be the least positive n such that every $P(n)$ is positive. We show that if X is a primitive set, then $g(X)\leqslant 2^k-2$. Moreover, there exists a primitive set Y such that $g(Y)=2^k-2$.

A matrix $A=(a_{ij})$ with real elements is called *nonnegative* $(A\geqslant 0)$ if $a_{ij}\geqslant 0$, and is called *positive* $(A>0)$ if $a_{ij}>0$ for all i, j.

A *primitive* matrix A is defined to be a $k\times k$ nonnegative matrix $(1<k<\infty)$ such that, for some positive integer n, $A^n>0$. Primitive matrices share important spectral and contractive properties with positive matrices and have been studied extensively [1].

A primitive matrix A has *index* g if $A^g>0$ but none of the matrices A^n, $0<n<g$, is positive. Wielandt [13] states without proof that, for any primitive matrix A, if $g(A)$ is the index of A, then

$$g(A)\leqslant k^2-2k+2, \tag{1}$$

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and the upper bound is attained by $g(B)$, where

$$B=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{2}$$

Moreover, Dionisio later showed that, up to a cogredient permutation of rows and columns, B is the unique primitive matrix with index k^2-2k+2, i.e., there are $k!$ distinct primitive matrices with index k^2-2k+2, each obtained from B by a cogredient permutation of rows and columns. (A matrix A is obtained by a cogredient permutation of a matrix E if, for some permutation matrix P, $A=PEP^T$.)

Proofs of these results, due to Rosenblatt, Holladay and Varga, Ptak, and Dionisio, are given by Seneta along with full citations [12, p. 54]; see [1, p. 48] for another presentation.

The bound (1) implies that to check whether a nonnegative matrix is primitive, one need compute at most its (k^2-2k+2)th power and check for positivity.

A natural generalization of a primitive matrix arises by replacing a single matrix and powers of that matrix with a set of matrices and inhomogeneous products of matrices from the set. Let X be a set of $k\times k$ nonnegative matrices. For a positive integer n, let $P(n)$ be an arbitrary product of n matrices from X, with any ordering and with repetitions permitted. Define X to be a *primitive set* if there is a positive integer n such that every $P(n)>0$. For any primitive set X of matrices, define the *index* $g(X)$ to be the least positive n such that every $P(n)>0$. Thus, if $g(X)=g$, there is at least one $P(g-1)$ that is not positive.

We shall show that if X is a primitive set, then

$$g(X)\leqslant 2^k-2. \tag{3}$$

Moreover, there exists a primitive set Y such that $g(Y)=2^k-2$.

This establishes for primitive sets the analogues of Wielandt's results for primitive matrices. Paz [11] has proved analogous results for sets of nonnegative $k\times k$ matrices, inhomogeneous products of which eventually become scrambling matrices.

We shall further show that the primitive sets at which (3) is an equality are not unique up to cogredient permutation of rows and columns, either of all matrices in the set simultaneously or of individual matrices independently.

A computational consequence of (3) is that, to check whether a given set X of matrices is primitive, one need check the positivity of products of at most $2^k - 2$ factors from X.

If $A \geqslant 0$, the *incidence matrix* $R(A)$ is the matrix obtained by replacing each positive element of A with 1 and leaving each 0 element as 0. If X is any set of nonnegative matrices, define $R(X) = \{R(A): A \in X\}$. Let J be the $k \times k$ matrix with every element 1. Then $A > 0$ if and only if $R(A) = J$. Also [2, p. 386], if $A \geqslant 0$, $E \geqslant 0$, then $(R(AE) = R(R(A)R(E))$. Thus the positivity of a product of nonnegative matrices depends only on the incidence matrices of the factors.

A primitive set is the combinatorial part of Hajnal's [9] concept of ergodic set, and is therefore relevant to the theories of inhomogeneous Markov chains [7, 8], products of random matrices [5], probabilistic automata [11], and weak ergodicity in mathematical demography [6, 10]. An *ergodic set* S is a primitive set such that there exists $r > 0$ and for every matrix A in S,

$$\frac{\min^+(A)}{\max(A)} \geqslant r, \tag{4}$$

where $\min^+(A)$ is the smallest among the positive elements of A, and $\max(A)$ is the largest element of A. Clearly, every finite primitive set, i.e., one containing only a finite number of distinct matrices, is an ergodic set.

Before proving (3) and showing that it gives the best possible bound, we give without proof some necessary and some sufficient conditions for a set of matrices to be primitive.

Let X be a primitive set. Then: $g(R(X)) = g(X)$; $g(X) \geqslant \max_{A \in X} g(A)$; for $n = 1, 2, \ldots$, $P(n)$ is primitive; $P(n) > 0$ for $n \geqslant g(X)$; and $g(P(n)) \leqslant$ the least integer greater than or equal to $g(X)/n$. Moreover, if M is any primitive square matrix of zeros and ones (of arbitrary order, not necessarily $k \times k$), and $M(X)$ is any matrix obtained from M by replacing each 0 element of M with a $k \times k$ 0 matrix and each 1 element of M with any $A \in X$ (the same or a different matrix may replace each different 1), then $M(X)$ is primitive, and $g(M(X)) \leqslant g(M)g(X)$. (This generalizes to inhomogeneous products a theorem [4] on multiregional age-structured populations.)

We now state some sufficient conditions. Let X be a set of $k \times k$ nonnegative matrices. X is primitive if there exists a primitive matrix E such that, for all A in X, $R(A) \geqslant R(E)$, in which case $g(X) \leqslant g(E)$ (Hajnal [9]); or if $X = \{A, J\}$, with A primitive, in which case $g(X) = g(A)$; or if the minimum of the row sums of every $R(A)$ in $R(X)$ and the minimum of the column sums of every $R(A)$ in $R(X)$ are each greater than $k/2$, in which case $g(X) \leqslant 2$; or if there exists a primitive set Y such that for every $A \in X$, there exists $E \in Y$ with $R(E) \leqslant R(A)$, in which case $g(X) \leqslant g(Y)$; or if, when S is any nonempty proper subset of $\{1,2,\ldots,k\}$, we have $\#\{j: a_{ij} > 0$ for some $i \in S\} > \#\{j: j \in S\}$, for any $A = \{A_{ij}\} \in X$, in which case $g(X) \leqslant k-1$. (This last result is due to the referee.)

THEOREM 1. *If X is a primitive set, then* (3) *holds.*

Proof. A $k \times k$ matrix B is positive if, and only if, for any nonzero $1 \times k$ matrix b of zeros and ones the product bB is positive. Accordingly, a set X of nonnegative $k \times k$ matrices is primitive if and only if there exists n such that, for any sequence $(B_1, B_2, \ldots, B_n)$ of elements of X and any $1 \times k$ matrix b of zeros and ones, the product $bB_1B_2 \cdots B_n$ is positive. Therefore, for a primitive set the following sequence S of incidence matrices always ends with the positive $1 \times k$ matrix $(1,1,\ldots,1)$:

$$S = (b, R(bB_1), R(bB_1B_2), \ldots, R(bB_1B_2 \cdots B_n)).$$

There are no repetitions among the terms of S up to the first occurrence of $(1,1,\ldots,1)$, for, if there were such a repetition, say

$$R(bB_1B_2 \cdots B_h) = R(bB_1B_2 \cdots B_i)$$

for $h < i$, then we could construct a product

$$bB_1B_2 \cdots B_h(B_{h+1}B_{h+2} \cdots B_i)^j$$

which would not be positive for any positive integer j. This is impossible for a primitive set.

The terms of S are nonzero $1 \times k$ matrices of zeros and ones. Since there are $2^k - 1$ such matrices, S has at most this many distinct terms. If the smallest possible n is used, that is, if $n = g(X)$, then for some b only the last term of S is equal to $(1,1,\ldots,1)$. Hence, all $n+1$ terms of S are distinct, and $n + 1 \leqslant 2^k - 1$, which means $g(X) \leqslant 2^k - 2$. ■

THEOREM 2. *There exists a primitive set Y such that* $g(Y) = 2^k - 2$.

Proof. Let $g = 2^k - 2$. We now construct a primitive set $Y = \{B_1, B_2, \ldots, B_g\}$ of index g, whose elements are $k \times k$ nonnegative matrices.

Let $(b_1, b_2, \ldots, b_{g+1})$ be a sequence without repetitions of all the nonzero $1 \times k$ matrices of zeros and ones, ordered so that the number of ones in the matrices is nondecreasing. Clearly, b_1 has a single 1 in it and $b_{g+1} = (1, 1, \ldots, 1)$.

Let us use the following notation:

$$b_i = (b_{i1}, b_{i2}, \ldots, b_{ik}),$$

$$b_{i+1} = \left(b_{(i+1)1}, b_{(i+1)2}, \ldots, b_{(i+1)k}\right),$$

$$[x \leqslant y] = \begin{cases} 1 & \text{if} \quad x \leqslant y, \\ 0 & \text{if} \quad x > y. \end{cases}$$

Then, for $i = 1, 2, \ldots, g$, define

$$B_i = \left(\left[b_{ip} \leqslant b_{(i+1)q}\right]\right)_{p,q},$$

where $p = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, k$.

First let us evaluate

$$b_i B_i = (b_{i1}, b_{i2}, \ldots, b_{ik})\left(\left[b_{ip} \leqslant b_{(i+1)q}\right]\right)_{p,q}.$$

This is a $1 \times k$ matrix whose qth element is

$$\sum_{p=1}^{k} b_{ip}\left[b_{ip} \leqslant b_{(i+1)q}\right].$$

It is evident that if $b_{(i+1)q} = 0$, this sum is zero, and if $b_{(i+1)q} = 1$, it is positive. Therefore, $R(b_i B_i) = b_{i+1}$, and

$$(b_1, b_2, \ldots, b_{g+1}) = \left(b_1, R(b_1 B_1), R(b_1 B_1 B_2), \ldots, R(b_1 B_1 B_2 \cdots B_g)\right).$$

This shows that Y has index g, provided it is a primitive set.

The following facts can be verified by a calculation which will be given at the end of this proof: For any nonzero $1 \times k$ matrix b_h of zeros and ones, and for any B_i, the number of positive terms in b_h is less than or equal to the number of positive terms in the product $b_h B_i$. Let $N(b)$ be the number of positive elements in a $1 \times k$ matrix $b \geqslant 0$. Then

$$N(b_h) \leqslant N(b_h B_i). \tag{5}$$

Furthermore, if equality holds in (5), then $h = i$ or $b_h = (1, 1, \ldots, 1)$.

These facts imply that Y is a primitive set. If it were not so, there would exist a nonzero $1 \times k$ matrix c of zeros and ones and an infinite sequence $(C_1, C_2, \ldots)$ of $k \times k$ matrices selected from Y such that the following infinite sequence T of $1 \times k$ matrices had at least one zero in every term:

$$T = (c, cC_1, cC_1C_2, cC_1C_2C_3, \ldots).$$

We know that

$$N(c) \leqslant N(cC_1) \leqslant N(cC_1C_2) \leqslant \cdots,$$

so that all except a finite number of $1 \times k$ matrices in the sequence T have exactly the same number of positive elements in them. Without loss of generality we may assume that

$$N(c) = N(cC_1) = N(cC_1C_2) = \cdots = m < k. \tag{6}$$

It has been stated that an equation such as $N(c) = N(cC_1)$ can only occur if it is of the form $N(b_i) = N(b_iB_i)$ for some i, and this is the same as $N(b_i) = N(b_{i+1})$. Therefore, (6) is of the form

$$N(b_i) = N(b_{i+1}) = N(b_{i+2}) = \cdots,$$

which is of length at most $\binom{k}{m}$. Therefore, the infinite sequence T does not exist. This proves that Y is a primitive set.

Now let us prove (5) and, if equality holds, $h = i$ or $b_h = (1, 1, \ldots, 1)$.

The $1 \times k$ matrix $b_h B_i$ has qth element

$$\sum_{p=1}^{k} b_{hp}\left[b_{ip} \leqslant b_{(i+1)q}\right]. \tag{7}$$

Let us consider two cases:

(i) Assume there is a p such that $b_{hp} = 1$ and $b_{ip} = 0$. Then (7) is positive for all q, which means that $N(b_hB_i) = k$, and accordingly $N(b_h) \leqslant N(b_hB_i)$ with equality only when $b_h = (1, 1, \ldots, 1)$.

(ii) Assume for all p that $b_{hp} \leqslant b_{ip}$. Since b_h is nonzero, there is an integer p such that $b_{hp} = 1$ and therefore $b_{ip} = 1$. For this value of p,

$$b_{hp}\left[b_{ip} \leqslant b_{(i+1)q}\right]$$

is positive if, and only if, $b_{(i+1)q}=1$. Also, (7) is positive if, and only if, $b_{(i+1)q}=1$. Hence

$$N(b_h B_i)=N(b_{i+1}).$$

This case is concluded as follows:

$$N(b_h)\leqslant N(b_i)\leqslant N(b_{i+1})=N(b_h B_i). \tag{8}$$

Finally, suppose $N(b_h)=N(b_h B_i)$. This cannot hold in case (i), unless $b_h=(1,1,\dots,1)$. In case (ii), (8) implies that $b_h=b_i$ and i must be one of those values for which $N(b_i)=N(b_{i+1})$. ■

To see that more than one primitive set has index 2^k-2, even allowing for cogredient permutations of the individual matrices, let $k=3$, $X_1=\{D,E\}$, $X_2=\{D,F\}$,

$$D=\begin{pmatrix}1&1&0\\0&0&1\\1&0&0\end{pmatrix},\qquad E=\begin{pmatrix}0&1&1\\1&1&0\\1&0&0\end{pmatrix},\qquad F=\begin{pmatrix}0&1&1\\1&0&1\\1&0&0\end{pmatrix}.$$

It may be checked that X_1 and X_2 are primitive and $g(X_1)=g(X_2)=6$. No cogredient permutation of E can produce F, because the third row of E must become the third row of F, and therefore in a cogredient permutation the third column of E must become the third column of F; but the third columns of E and F differ.

These examples raise an interesting unanswered question: What is the smallest number of matrices in a primitive set Y such that $g(Y)=2^k-2$? When $k=3$, the answer is 2.

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